# Local anisotropic effects on multifractality of turbulence

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(Received 11 February 1993)

We study the differences in multifractal properties of rate of dissipation and related fields such as $du^2/dt$, $(\partial u/\partial t)^2$, and $(\partial u/\partial x)^2$ as well as enstrophy and enstrophy generation. It is argued that these differences can result from local anisotropy of the flow properties in subregions with high values of corresponding fields. Local "cascades" of such quantities as helicity and inviscid helicity generation are considered in this connection. A number of results from laboratory experiments and field observations are analyzed in the context of the above differences and related problems.

PACS number(s): 47.27.−i

## I. MOTIVATION

The rate of energy dissipation in flows of a Newtonian fluid is given by the following expression:

$$\varepsilon = 2\nu s_{ij} s_{ij} , \tag{1}$$

where

$$s_{ij} = \tfrac{1}{2}(\partial u_i/\partial x_j + \partial u_j/\partial x_i) ,$$

and is a spatially distributed and time-dependent stochastic field. It is related to other flow fields by simple relations in the mean

$$\langle \varepsilon \rangle = \left\langle \left| \frac{du^2}{dt} \right| \right\rangle = 2\nu \langle \omega^2 \rangle \tag{2}$$

under rather general conditions. Here, $\boldsymbol{\omega} = \text{curl}\mathbf{u}$ is the field of vorticity. In locally isotropic turbulence there exists an additional relation

$$\langle \varepsilon \rangle = 15\nu \left\langle \left[ \frac{\partial u_x}{\partial x} \right]^2 \right\rangle , \tag{3}$$

where $x$ is some fixed axis (usually in the direction of the mean flow).

It is also widely accepted to use the Taylor hypothesis for interpretation of experimental data so that, in addition to (1)–(3), one has the relation

$$\langle \varepsilon \rangle = 15 U^{-2} \nu \left\langle \left[ \frac{\partial u_x}{\partial t} \right]^2 \right\rangle , \tag{4}$$

where $U$ is the (local) mean velocity.

Of course, it does not follow from (2)–(4) that other properties of these fields should be the same or even similar [1–5]. In this paper we study the differences in multifractal properties of the above-mentioned fields and their relation to cascade processes (i.e., dynamical scale splitting) of a different nature.

## II. KOLMOGOROV TURBULENCE

Hypotheses of Kolmogorov type allow one to obtain scaling laws in the inertial range from dimensional arguments. However, these arguments are, generally speaking, inapplicable for determining the generalized dimensions $D_q$ in the case when one is interested in multifractal properties of some field in turbulent flow $\chi$ (see, for example, [5]). The generalized dimension $D_q$ of the field $\chi$ is introduced in the following way:

$$\langle \chi_r^q \rangle = N^{-1} \sum_i^N \left[ r^{-3} \int_{v_i(r)} \chi \, dv \right]^q \sim r^{-\mu_q} , \tag{5}$$

$$\mu_q = (3 - D_q)(q-1) , \tag{6}$$

where $v_i(r)$ are volumes of scale $r$ covering the flow region and $N$ is the number of such volumes. The representation (5) is a scaling hypothesis for some range of scales. In the case $D_q \neq \text{const}$, it is questionable whether dimensional arguments can be used for finding $D_q$ (or $\mu_q$). This problem is closely related to the hypothesis on scaling behavior of quantities like [5]

$$\int_{v(r)} \chi \, dv \sim r^{\alpha} . \tag{7}$$

The exponent $\alpha$ in (7), generally, is a statistical variable, i.e., it is not a fixed quantity, and therefore cannot be found from dimensional arguments. However, if there exists a limit

$$\lim_{q \to \infty} D_q = D_\infty ,$$

it is easy to show that [6]

$$\max_i \left\{ \int_{v_i(r)} \chi \, dv \right\} \sim r^{D_\infty} . \tag{8}$$

Since $D_\infty$ is a constant, dimensional arguments *may* be applicable for its determination.

In Kolmogorov turbulence the only governing dimensional parameter in the inertial range is $\langle \varepsilon \rangle$. Applying dimensional arguments to (8) and the field $\chi = \partial u^2/\partial t$, it is straightforward to obtain that $D_\infty = 3$, i.e., this field is monofractal ($q \geq 0$). The *same* arguments for the field $\chi = (\partial u/\partial t)^2$ give $D_\infty = \frac{7}{3}$, and for the other three fields $(\partial u/\partial x)^2$, normalized dissipation $s_{ij}s_{ij}$, and enstrophy $\omega^2$, the value $D_\infty = \frac{5}{3}$. It is noteworthy that the value $D_\infty = 3$

 

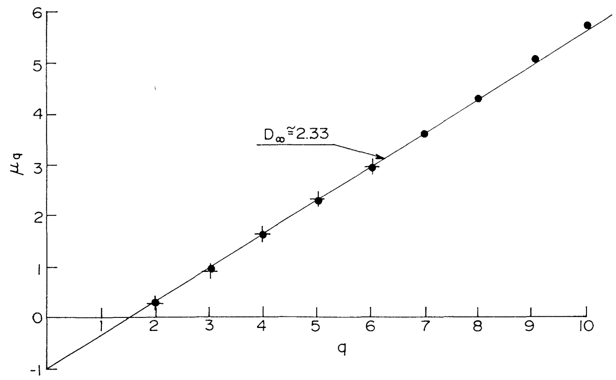


FIG. 1. Intermittency exponent $\mu_q$ for the field $(\partial u/\partial t)^2$ in the atmospheric surface layer [11] and turbulent grid flow [2].

for the field $\chi=\partial u^2/\partial t$ has been obtained from different arguments in [7]; the value $D_\infty=\frac{7}{3}$ is discussed in [6], while the value $D_\infty=\frac{5}{3}$ for the fields $s_{ij}s_{ij}$ and $\omega^2$ can be related to the results of [8–10].

Experimentally, the asymptotic behavior of $D_q$ for large $q$ (i.e., $D_\infty$) can be found by means of the following relation:

$$\mu_q \sim (3-D_\infty)q \ . \tag{9}$$

Experimental results on $\mu_q$ shown in Fig. 1 for the field $(\partial u/\partial t)^2$ correspond to experiments in the atmospheric surface layer [11] ($\mathrm{Re}_\lambda \sim 10^4$) and laboratory turbulent grid flow [2] ($\mathrm{Re}_\lambda \sim 70$). It is seen quite clearly that in spite of a large difference in Reynolds number, the results of both experiments are very close and exhibit a fast approach to the asymptotic behavior (9) starting from $q=2$ with $D_\infty=\frac{7}{3}$, in full agreement with the value obtained above for Kolmogorov turbulence (see also [6]). However, this is quite surprising in view of the following three circumstances: (i) It is unlikely that turbulent grid flow at such low Reynolds numbers will possess the properties of Kolmogorov turbulence, (ii) it is generally accepted that in such experiments $(\partial u/\partial x)^2$ is measured (the Taylor hypothesis) but $(\partial u/\partial t)^2$, for which $D_\infty$ is different for Kolmogorov turbulence, and (iii) the fast approach of the experimental data of the asymptotics (9) is an indication of strong spatial localization of the field $(\partial u/\partial x)^2$. For these reasons it is unlikely that we are dealing here with the properties of Kolmogorov turbulence.

### III. LOCAL "CASCADE" OF HELICITY

There is a substantial amount of evidence from laboratory and numerical experiments that in turbulent flows there exist regions of concentrated vorticity, which most probably take the form of thin vortex tubes—filaments [12–17,21,26,29]. Theoretical considerations also indicate clearly that such regions of concentrated vorticity should exist in turbulent flows (see, for example [18,19]).

Inconsistencies have been revealed also within the multifractal ideology (which is based on the Kolmogorov cascade) and it has been suggested [20] that these inconsistencies can be resolved by an assumption of the tendency of turbulent flows towards a local two-dimensional state in some sense in small scales (see [21,22] and references therein). Also, the three circumstances mentioned in Sec. II seem to be related to this tendency. Indeed, for example, in [23] a possibility of a helicity cascade is investigated as an alternative to the cascade of energy in three-dimensional turbulence. The possibility of a helicity cascade is rooted in the fact that helicity (along with kinetic energy) is an inviscid invariant of fluid motion (see, for example [24]):

$$\frac{d\mathcal{H}}{dt}=\oint_S \omega_n Q_n ds - \nu \int_V \boldsymbol{\omega}\cdot\mathrm{curl}\boldsymbol{\omega}\, dv \ , \tag{10}$$

where

$$\mathcal{H}=\int_V h\, dv \ , \quad h=\mathbf{u}\cdot\boldsymbol{\omega} \ , \quad Q=\tfrac{1}{2}u^2-p/\rho \ , \tag{11}$$

$S$ is the surface bounding some volume $V$, $p$ is the pressure, $\rho$ is the fluid density, and the fluid is assumed incompressible.

A similar relation is valid for kinetic energy,

$$d\left[\int_V u^2 dv\right]\Big/dt=-\oint_S p u_n ds-\int_V \varepsilon\, dv \ . \tag{12}$$

The main ingredient of the cascade ideology is the assumption that in the limit $\nu \to 0$,

$$0<\lim_{\nu\to 0}\left[\int_V \varepsilon\, dv\right]<\infty \ . \tag{13}$$

In order to apply this ideology to helicity, the surface integral in (10) is, generally, supposed to be vanishing, assuming that

$$0<\lim_{\nu\to 0}\left[\nu\left|\int_V \boldsymbol{\omega}\cdot\mathrm{curl}\boldsymbol{\omega}\, dv\right|\right]<\infty \ . \tag{14}$$

It is important that in the case of helicity (in contradistinction to the case of energy; see below), there exists a possibility of an "inviscid" cascade. This is due to the surface integral in (10) containing $\omega_n$. Indeed, let us write down (10) for a vortex tube neglecting viscosity and taking into account that $\omega_n=0$ at the lateral surface of the vortex tube

$$\frac{d\mathcal{H}}{dt}=\oint_{S_1}\omega_n Q\, ds+\oint_{S_2}\omega_n Q\, ds \ , \tag{15}$$

where $S_1$ and $S_2$ are the surfaces of the vortex-tube cross sections. Suppose that statistically the dynamics of vortex tubes (which is predominantly stretching) is such that $S_1,S_2 \to 0$ and there exists a finite limit

$$0<\lim_{S_1,S_2\to 0}\left[\left|\oint_{S_1}\omega_n Q\, ds+\oint_{S_2}\omega_n Q\, ds\right|\right]<\infty \ . \tag{16}$$

This means that there exists an inviscid cascade of helicity on the set of vortex tubes (filaments). The meaning of the condition (16) consists of three important aspects. First, it means that in the limit vortex tubes turn into vortex filaments *of finite intensity*, i.e., $\omega_n$ in the vortex-tube cross sections becomes a $\delta$ function of the area of the cross section. The second aspect is that the limiting "length" of vortex tubes remains finite and smaller than

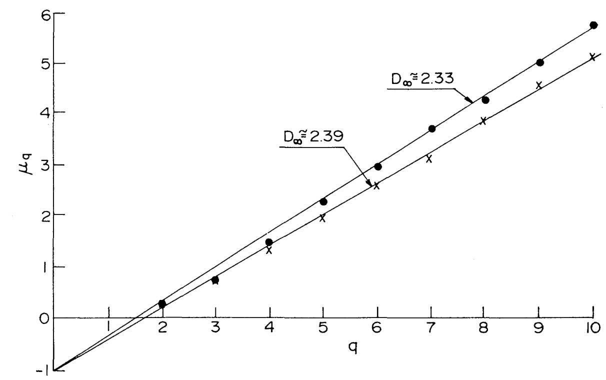


FIG. 2. Intermittency exponent $\mu_q$ for enstrophy $\omega^2$ in turbulent grid flow [2] (●) and at the axis of a turbulent jet [25] (×).

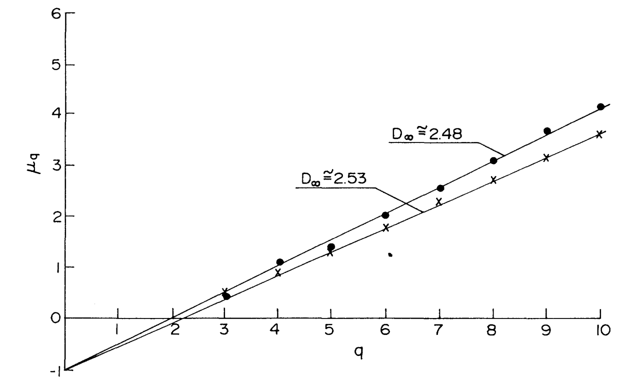


FIG. 4. Intermittency exponent $\mu_q$ of dissipation in grid flow [2] and in a jet [25].

the integral (correlation) scale of turbulent flow, since otherwise the limit in (16) would vanish (in the mean). Finally, the third aspect is that the relation (16) means that in the process of the stretching of vortex tubes, the surface integral in (10) does not vanish when the volume $V \to 0$, while the surface integral in (12) is vanishing when $V \to 0$.

The environment of vortex tubes, which is predominantly stretching them, plays the role of source (sink) of helicity and maintains the cascade, i.e., the dynamic splitting of scales. It is also noteworthy that this cascade is, roughly speaking, localized on the set of vortex tubes. Also this cascade is not connected directly with the action of viscosity. Therefore, it is plausible that it will preponderate over the cascade of energy (which is related directly to the balance of viscous forces) on the set of vortex tubes. Since the inviscid cascade is localized, it will hardly have any influence on such characteristics as energy spectra. However, the situation is different in case of high moments of some quantity $\chi$, which by definition [see (5)] single out the subregions with high concentration of $\chi$. Therefore, the behavior of high moments of $\chi$ should be governed by the dynamics of such subregions. In particular, if smoothed maxima [see (8)] of such fields as $(\partial u/\partial x)^2$ or $\omega^2$ are located in the proximity of vortex tubes, it is natural to expect that $D_\infty$ for such fields will be controlled in the inertial range not by $\langle \varepsilon \rangle$ but rather by the parameter $\langle |dh/dt| \rangle$. Thus it follows from dimensional arguments that

$$\max_i \left\{ \int_{v_i} \left[ \frac{\partial u}{\partial x} \right]^2 dv \right\} \sim \left\langle \left| \frac{dh}{dt} \right| \right\rangle^{2/3} r^{7/3} , \tag{17}$$

and in a similar way for the fields of enstrophy,

$$\max_i \left\{ \int_{v_i} \omega^2 dv \right\} \sim \left\langle \left| \frac{dh}{dt} \right| \right\rangle^{2/3} r^{7/3} , \tag{18}$$

and dissipation,

$$\max_i \left\{ \int_{v_i} s_{ij} s_{ij} dv \right\} \sim \left\langle \left| \frac{dh}{dt} \right| \right\rangle^{2/3} r^{7/3} . \tag{19}$$

Thus for all the three fields $(\partial u/\partial x)^2$, $\omega^2$, and $s_{ij}s_{ij}$, the dimension $D_\infty = \frac{7}{3}$. Hence, the results shown in Fig. 1 can be explained by means of relation (17), which seems also to remove the three questions posed at the end of Sec. II. In order to check the representation (18) for the enstrophy field, we show the relations for $\mu_q$ (Fig. 2) obtained from the same experimental results for grid flow [2] (as in Fig. 1) and at the center of turbulent circular jet [25] with $\mathrm{Re}_\lambda = 880$. In the first case, $D_\infty \cong 2.33$ and in the second, $D_\infty \cong 2.39$, i.e., both values are in good agreement with the representation (18).

In the case of the field of dissipation, the situation is more complicated. On the one hand, the results of direct numerical simulations [26] are in good agreement with the asymptotic behavior (19), as can be seen from Fig. 3. On the other hand, the results of experiments used above [2,25] give for both flows $D_\infty \cong 2.5$ (see Fig. 4), which is different both from the value given by the relation (19) and the results of numerical simulations [26]. This seemingly contradictory situation can be resolved via consideration of the instability of the process of filamentation and a new type of cascade related to it.

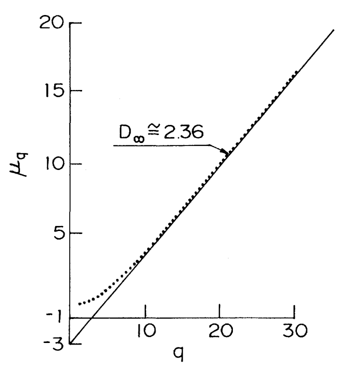


FIG. 3. Intermittency exponent $\mu_q$ of dissipation from a direct numerical simulation [26].

## IV. LOCAL "CASCADE" OF INVISCID HELICITY GENERATION

The very fact that in experiments with turbulent grid flow and turbulent jet [2,25] the values of $D_\infty$ are different for the fields $(\partial u/\partial x)^2$, $\omega^2$ and the field of dissipation $s_{ij}s_{ij}$ is an indication of flow anisotropy in subregions with large dissipation. In order to provide a qualitative and a quantitative explanation of the contradiction mentioned at the end of Sec. III, let us introduce a local representation of the field $s_{ij}$ in a system of reference with the axis $x_1$ coinciding with the axis of the vortex filament and the following substitution:

$$\hat{s}_1 = s_{23}, \quad \hat{s}_2 = s_{13}, \quad \hat{s}_3 = s_{12}, \quad \hat{s}_4 = \tfrac{1}{2}(s_{11} - s_{22}) ,$$

$$\hat{s}_5 = \frac{1}{2\sqrt{3}}(s_{11} + s_{22} - 2s_{33}) ,$$

$$\hat{s}_6 \sim (s_{11} + s_{22} + s_{33}) = 0 .$$

In these variables

$$\tfrac{1}{2} s_{ij}s_{ij} = \sum_{i=1}^{5} \hat{s}_i^2 . \tag{20}$$

The components $\hat{s}_i$ form three subgroups realizing different representations of the axial symmetry group on disturbances in the form

$$\mathbf{u} \sim \exp\{im(\phi - ct)\}\mathbf{g}(x_1, r)$$

in a cylindrical system of reference with axis $x_1$ along the axis of the vortex filament. Namely, $\hat{s}_5$ realizes the representation with $m=0$, the subgroup $\hat{s}_2$ and $\hat{s}_3$ with $m=1$ (single helix), and the subgroup $\hat{s}_1$ and $\hat{s}_4$ with $m=2$ (double helix). Disturbances with different $m$ have different properties. For example, a simple model of stability of a vortex tube has been given in [27] and it has been shown that the modes with $m=1$ are the most dangerous ones in the case of three-dimensional instability (see, also, [28]). It has been suggested that these modes "might provide the dominant three-dimensional components to the splitting process" [27]. In such a case, the main contribution to $s_{ij}s_{ij}$ in (20) will be made (statistically) by the components $\hat{s}_2$ and $\hat{s}_3$ (i.e., $s_{13}$ and $s_{12}$), while the components $s_{11}$, $s_{22}$, and $s_{33}$ will have little influence (again statistically) on $s_{ij}s_{ij}$ in the regions of vortex-tube instability. Moreover, in experiments with mean velocity much larger than turbulence intensity, streamwise vortices are in contact with a stationary probe for a much longer time than arbitrarily oriented vortex filaments. Therefore, it is natural to expect the "group preference" of $s_{12}$ and $s_{13}$ in $s_{ij}s_{ij}$ with the axis $x_1$ in the streamwise direction. In other words, $(\partial u/\partial x)^2$ will make a small contribution to

$$\max_i \left\{ \int_{v_i} s_{ij}s_{ij}\, dv \right\} ,$$

i.e., this last quantity, and

$$\max_i \left\{ \int_{v_i} \left[\frac{\partial u}{\partial x}\right]^2 dv \right\}$$

may be separated in space and have essentially different properties in such experiments.

The right term in the equation for helicity density [24] [neglecting viscosity; for notations see (10) and (11)],

$$\frac{dh}{dt} = \mathrm{div}(\boldsymbol{\omega} Q) , \tag{21}$$

can be interpreted as an inviscid helicity generation term. It is plausible that in helical instability on vortex tubes (filaments) this quantity should play an essential role. It is easy to show that for a vortex tube,

$$\frac{d}{dt}\left\{ \int_V \mathrm{div}(\boldsymbol{\omega} Q)\, dv \right\} = \oint_{S_1} \omega_n Q'\, ds + \oint_{S_2} \omega_n Q'\, ds + \cdots , \tag{22}$$

where

$$Q' = \frac{\partial Q}{\partial t} + u_n \frac{\partial Q}{\partial n} .$$

Comparing (22) with (15), one can assume the possibility of a local cascade of inviscid helicity generation $\mathrm{div}(\boldsymbol{\omega} Q)$ on vortex tubes in full analogy with the local cascade of helicity discussed in Sec. III. In the present case the quantity $Q$ in the condition (16) should be replaced by $Q'$. In a similar way the environment of vortex tubes plays the role of a source (sink) of the inviscid helicity generation and maintains the local cascade. The parameter governing the scaling processes in such a situation would be the quantity

$$G = \left| \left\langle \frac{d\{\mathrm{div}(\boldsymbol{\omega} Q)\}}{dt} \right\rangle \right| . \tag{23}$$

It follows then from dimensional arguments that

$$\max_i \left\{ \int_{v_i} s_{ij}s_{ij}\, dv \right\} \sim G^{1/2} r^{5/2} , \tag{24}$$

i.e., in this case the dimension $D_\infty = \frac{5}{2}$ for the field $s_{ij}s_{ij}$. This is in good agreement with the experimental data for the grid turbulent flow and for the turbulent jet discussed above (see Fig. 4).

There are clear indications from direct numerical simulations [29] that the maxima of the enstrophy generation term

$$\max_i \left| \int_{v_i} \omega_j \omega_k s_{jk}\, dv \right|$$

are also realized in the regions of instability of the vortex tubes (filaments). Additional arguments are given below. In our context this means that $D_\infty$ can be found using as a governing parameter the inviscid helicity generation $G$ as defined by (23). In this case the result is as follows:

$$\max_i \left| \int_{v_i} \omega_j \omega_k s_{jk}\, dv \right| \sim G^{3/4} r^{9/4} , \tag{25}$$

i.e., in this case, $D_\infty = \frac{9}{4}$. The value obtained from experimental data for the flow past a grid and jet is $D_\infty \simeq 2.19$, i.e., rather close to the value obtained from (25). This is seen from Fig. 5 in which the intermittency exponents $\mu_q$

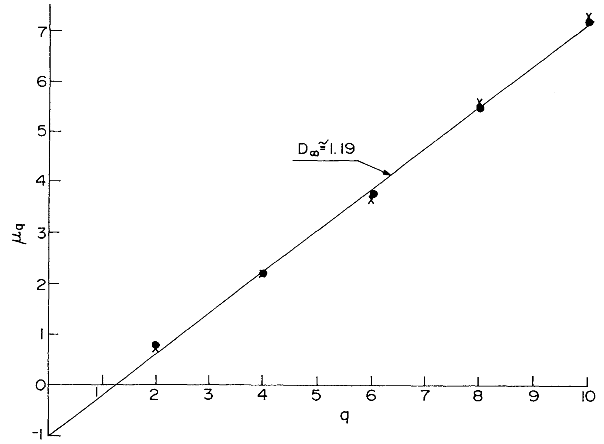


FIG. 5. Intermittency exponent $\mu_q$ (even $q$) for the enstrophy generation term $\omega_i \omega_j s_{ij}$ in grid flow [2] and in a jet [25].

are presented for both flows for the quantity $\omega_j \omega_k s_{jk}$.

The above agreement between the values of $D_\infty$ obtained from experiments and from (25) can be interpreted as an indication that values

$$\max_i \left| \int_{v_i} \omega_j \omega_k s_{jk} dv \right|$$

(on the scales from the inertial range) are located in the subregions of instability of the vortex tubes (filaments). Note that $D_\infty$ for the field $|(\partial u/\partial x)^3|$ is equal to $\frac{7}{3}$. This is different from the value of $D_\infty = \frac{9}{4}$ for the field $\omega_i \omega_j s_{ij}$.

The "maximization" of

$$\left| \int_{v_i} \omega_j \omega_k s_{jk} \right|$$

in subregions of dynamical instability is likely due to the property of such regions to be the source of dynamical irreversibility in turbulent flows. (Note that the scalar $\omega_j \omega_k s_{kj}$ changes its sign when the time is reversed; see also [30,31].) This can be seen from the following arguments: Let us introduce a (localized) probability density distribution $p(\eta)$ via

$$< \frac{1}{v} \int_v \omega_j \omega_k s_{jk} dv > = \int \eta p(\eta) d\eta \ , \quad (26)$$

where $< >$ has the meaning of a local ensemble average over the regions with largest

$$\left| \frac{1}{v} \int_v \omega_j \omega_k s_{jk} dv \right| .$$

Representing $p(\eta)$ as a sum of symmetric and antisymmetric parts

$$p(\eta) = p_+(\eta) + p_-(\eta) \ ,$$

where

$$p_+(\eta) = \{p(\eta) + p(-\eta)\}/2$$

and

$$p_-(\eta) = \{p(\eta) - (-\eta)\}/2 \ ,$$

the relation (26) takes the form

$$< \frac{1}{v} \int_v \omega_j \omega_k s_{jk} dv > = \int \eta p_-(\eta) d\eta \ . \quad (27)$$

If the motion is statistically reversible in time, then $p_-(\eta) \equiv 0$ and therefore

$$\frac{1}{v} < \int_v \omega_j \omega_k s_{jk} > \equiv 0 \ .$$

It is seen from this speculation that subregions with $\max_i \int_{v_i} \omega_j \omega_k s_{jk}$ should be looked for among the subregions with "maximal irreversibility", i.e., subregions with local dynamical instability in a stochastic flow.

## ACKNOWLEDGMENTS

The authors are grateful to E. Kit and M. Spector for stimulating discussions, I. Procaccia for sending us Ref. [20] before publication, and H. Vaisburd for essential help in data processing.


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